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Parameterized Joint Densities with Gaussian Mixture Marginals and their Potential Use in Nonlinear Robust Estimation

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Abstract— **This paper addresses the challenges of the fusion of two random vectors with imprecisely known stochastic dependency. This problem mainly occurs in decentralized estimation, e.g. of a distributed phenomenon, where the stochastic dependencies between the individual states are not stored. To cope with such problems we propose to exploit parameterized joint densities with both Gaussian marginals and Gaussian mixture marginals. Under structural assumptions these parameterized joint densities contain all information about the stochastic dependencies between their marginal densities in terms of a *generalized correlation parameter vector* $\underline{\xi}$. The parameterized joint densities are applied to the prediction step and the measurement step under imprecisely known correlation leading to a whole family of possible estimation results. The resulting density functions are characterized by the *generalized correlation parameter vector* $\underline{\xi}$. Once this structure and the bounds of these parameters are known, it is possible to find bounding densities containing all possible density functions, i.e., conservative estimation results.**

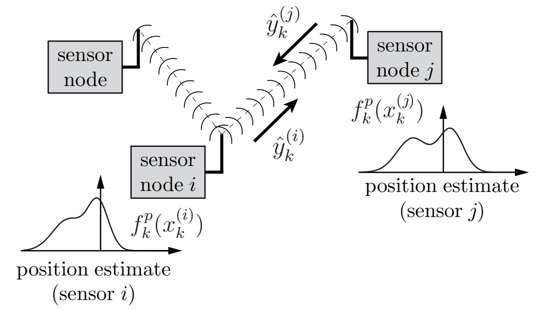

Fig. 1. Simple example scenario for the self-localization of a sensor-actuator-network. The position estimates $f_k^p(x_k^{(i)})$ and $f_k^p(x_k^{(j)})$ of the individual sensor nodes are improved by means of the distance measurements $\hat{y}_k^{(i)}$ and $\hat{y}_k^{(j)}$.

I. INTRODUCTION

In technical systems the state estimation by fusing uncertain information is a topic of extraordinary importance, e.g., observation of a distributed phenomenon by means of a sensor network [1], multiple target tracking [2], and robot localization [3]. For such applications, usually the unobservable internal state has to be reconstructed on the basis of an appropriate system model together with a stochastic noise model. In most cases, the Kalman filter and its many variations have proven to be useful [4].

Problems mainly occur when the stochastic dependencies between the states and/or the sensor noises are not precisely known, i.e., their joint statistics are simply not available. In that case, classical filtering techniques like the Kalman filter conveniently assume stochastic independence or known correlation, which automatically leads to unjustified improvement of estimation results. For understanding the source of unknown stochastic dependency a typical application is considered: Decentralized self-localization of a sensor-actuator-network, see Fig. 1.

In this scenario there exists a cause for stochastic dependency common to all distributed estimation processes, namely sharing of information among decentralized estimators. For example, the distance measurements $\hat{y}_k^{(i)}$ and $\hat{y}_k^{(j)}$ are used to improve the position estimate of two sensor nodes. In the linear case, using a centralized approach, the position of the individual sensor nodes can be estimated easily by applying a Kalman filter to the augmented state vector containing the positions of *all* sensor nodes. In that case, the estimator stores the associated stochastic dependencies and uses them for the next estimation step. However, for practical applications, especially for large sensor-actuator-networks, it is often desirable to reduce the heavy computational burden and to reduce communication activities between the individual nodes to a minimum. This leads to a decentralized estimation approach implying that just parts of the state vector are manipulated at each update step. Unfortunately, applying the Kalman filter while ignoring the existing dependencies between the individual states leads to overoptimistic estimation results. Coping with such problems is one of the main justificications for robust filter design [3]. The comparison between a centralized and decentralized estimator is visualized in Fig. 2 (a) and (b).

To tackle the previously mentioned problems two robust estimators have been introduced, namely Covariance Intersection [5], [6] and Covariance Bounds [7]. Basically, the resulting filters do not neglect the unknown stochastic dependency, but consider them by producing conservative estimates compatible with all correlations within an assumed correlation structure. Besides, the generalization of the Covariance Intersection based on a minimization of the Chernoff information is worthwhile mentioning [8].

Robust filters based on Covariance Intersection and Covariance Bounds rely on the correlation coefficient r, which is a sufficient measure for the stochastic dependency of Gaussian densities. That means, although these filters are efficient for linear state-space models and linear measurement models, they cannot be directly applied to nonlinear models, e.g., of a distributed phenomenon or nonlinear measurement equation. In addition, they are not able to work with more

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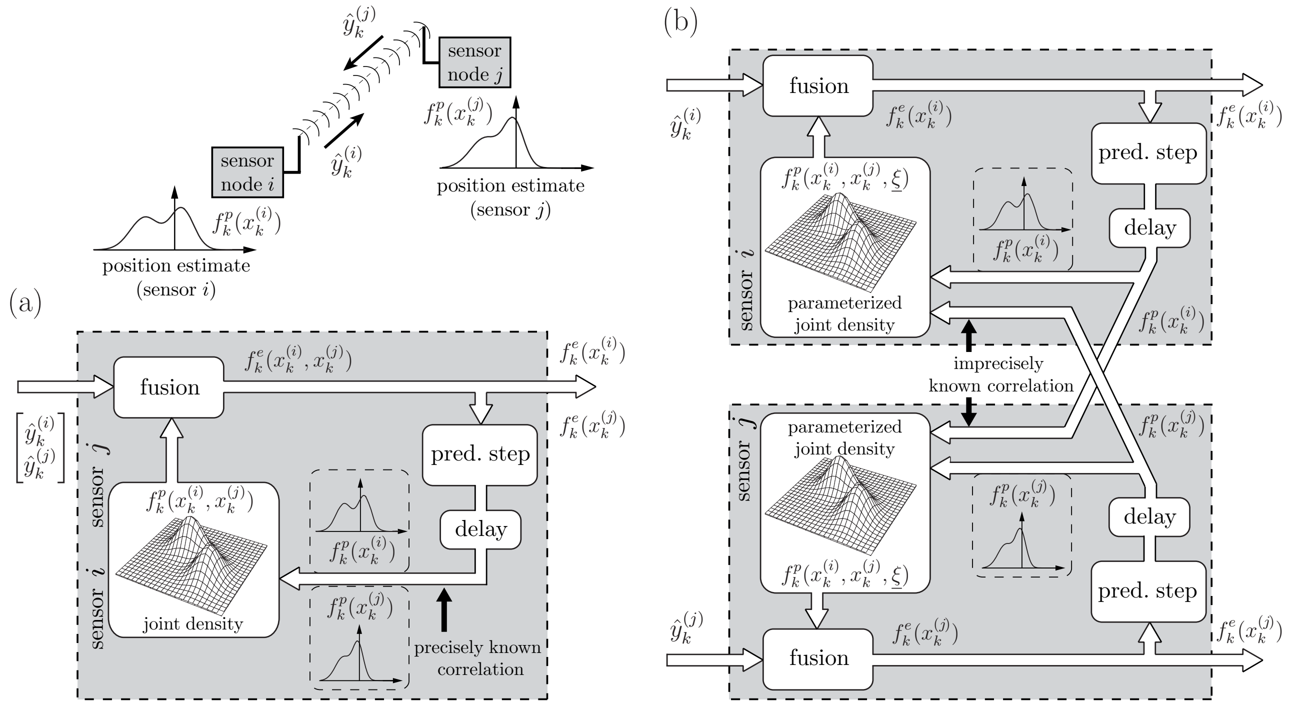

Fig. 2. **(a) Centralized estimation:** Since the estimation result is derived centrally the stochastic dependency between the estimate of sensor i and sensor j, and thus the joint density $f_k^p(x_k^{(i)}, x_k^{(j)})$ is *precisely* known. **(b) Decentralized estimation:** Due to the *imprecisely known* stochastic dependency between the estimate of Sensor i and Sensor j, the joint density $f_k^p(x_k^{(i)}, x_k^{(j)})$ is *not precisely* known, although the marginal densities $f_k^p(x_k^{(i)})$ and $f_k^p(x_k^{(j)})$ are given.

complex density functions such as Gaussian mixtures, which are known as universal approximators and thus, well-suited for nonlinear estimation problems [9].

According to previous discussions, one of the main mathematical challenges of a robust decentralized estimation based on more complex density functions, such as Gaussian mixtures, is that for these functions the *classical* correlation coefficient r is not a sufficient measure for the stochastic dependency. This implies that for such density functions no obvious correlation coefficient exists, which can be bounded and thus, no bounding density can be found in the classical sense.

In our previous research work [10] *parameterized joint densities* were derived with both Gaussian marginals and Gaussian mixture marginals. These parameterized joint densities contain all information about the stochastic dependency between their marginal density functions in terms of a parameter vector $\underline{\xi}$. This parameter vector can be regarded as a kind of *generalized correlation parameter* for the assumed structure of stochastic dependency.

In this paper, we are exploiting the parameterized joint densities for the prediction step and the measurement step under imprecisely known correlation. This leads to a whole family of possible estimation results characterized by the *generalized correlation parameter vector* $\underline{\xi}$. Furthermore, it is shown that the so-called parameterized estimation result can be bounded possibly by means of bounding densities, similar to [11]. Thus, when the structure and the bound of these correlation parameter vectors $\underline{\xi}$ are known, bounding densities which are compatible with all stochastic dependency structures can be found.

The remainder of this paper is structured as follows. In Section II, a rigorous formulation of the problem of state estimation with imprecisely known correlations is given. Section II then derives two special types of parameterized joint densities both with Gaussian marginals and Gaussian mixture marginals. In Section IV and section V, their potential use for robust nonlinear estimation is presented, i.e., prediction step and mesurement step. Furthermore, it will be shown that bounding the parameters of these joint densities, it is possible to find conservative estimation results even for nonlinear problems and for more complex density functions, such as Gaussian mixtures.

Throughout this paper, we use the notation $\mathcal{N}(\underline{z}, \mathbf{C}(r))$ to represent a two-dimensional Gaussian density, which is defined as

$$\mathcal{N}(\underline{z}, \mathbf{C}(r)) = \frac{1}{2\pi\sqrt{|\mathbf{C}(r)|}} \exp\left\{-\frac{1}{2}\underline{z}^T \mathbf{C}(r)^{-1} \underline{z}\right\} \quad ,$$

where

$$\underline{z} = \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \quad , \qquad \mathbf{C}(r) = \begin{bmatrix} C_x & r\sqrt{C_x C_y} \\ r\sqrt{C_y C_x} & C_y \end{bmatrix} \quad ,$$

are state vector and covariance matrix, respectively. The expected value of the states are denoted by $\hat{x}$ and $\hat{y}$, and $r \in [-1, 1]$ denotes the *classical* correlation coefficient.

II. PROBLEM FORMULATION

As mentioned in the introduction, there are many sources of stochastic dependencies. In this section, we take up the previous mentioned example, the decentralized self-localization of a sensor-actuator-network. By this means we are able to clarify the main problem common to all sources of imprecisely known stochastic dependency: *imprecisely* known joint densities with given marginal densities.

It is assumed that the distance measurement $\hat{y}_k$ is related *non*linearly to the position of sensor node i and sensor node j according to

$$\hat{y}_k = h_k(\underline{x}_k) + v_k \ ,$$

where the state vector $\underline{x}_k = [x_k^{(i)}, x_k^{(j)}]^T$ contains the individual scalar sensor positions and v_k represents the measurement uncertainty or model parameter uncertainties. The corresponding density functions of the sensor positions are given by $f_k^p(x_k^{(i)})$ and $f_k^p(x_k^{(j)})$, respectively.

In the additive noise case, the posterior density f_k^e can be easily calculated based on the prediction result f_k^p (prior density) according to

$$f_k^e(\underline{x}_k) = c_k^e \, f_k^L\left(\hat{y}_k - h_k(\underline{x}_k)\right) \, f_k^p(x_k^{(i)}, x_k^{(j)}) \ , \qquad (1)$$

where c_k^e is a normalization constant and f_k^L is the likelihood function.

With the justification of the considered example scenario we assume there exists an *imprecisely known* stochastic dependency between the individual estimates $\underline{x}_k = [x_k^{(i)}, x_k^{(j)}]^T$. That means although the individual marginal densities $f_k^p(x_k^{(i)})$ and $f_k^p(x_k^{(j)})$ are given, the joint density $f_k^p(x_k^{(i)}, x_k^{(j)})$ with all its information about the stochastic dependency is not precisely known. However, as it can be seen in (1), the knowledge of the joint density or at least its parameterization in terms of a correlation parameter is essential for the estimation process. Fig. 2 (b) illustrates the decentralized estimation process and shows the necessity of the joint density $f_k^p(x_k^{(i)}, x_k^{(j)})$.

If the joint density, and thus, the correlation structure would be known precisely, this correlation could be tracked easily and be considered in the next processing step. However, when the correlation structure is not precisely known, the joint densities need to be reconstructed somehow.

In the next section, the reconstruction of joint densities $f(x, y)$ based on known marginal densities $f_x(x)$ and $f_y(y)$ is discussed in more detail. For all introduced types this leads to a *parameterized joint density* depending on a generalized correlation function $\xi(r)$ or generalized correlation parameter vector $\underline{\xi}$, which contains the information about the stochastic dependency between the considered random variables.

III. PARAMETERIZED JOINT DENSITIES

A. Mixtures of Correlated Jointly Gaussian Densities

In this section, we present a parameterized joint density based on the integral of jointly Gaussian densities with different *classical correlation coefficients* r. The weighting factors of the individual joint densities need to be chosen in such a way that the marginals are represented by the given

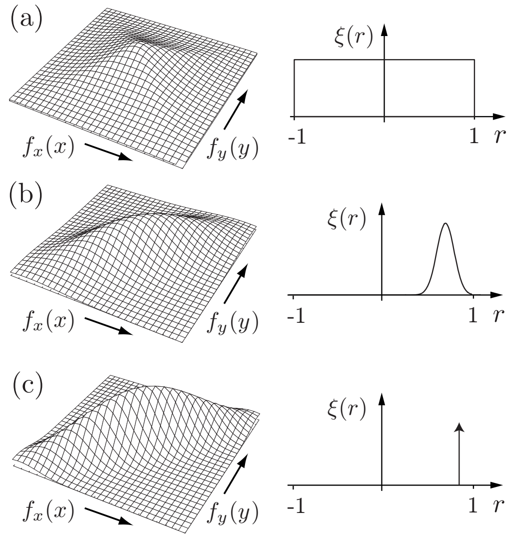

Fig. 3. Joint density consisting of the integral of correlated Gaussian densities for different generalized correlation functions $\xi(r)$, **(a)** $\xi(r) = 0.5$, **(b)** $\xi(r) = \mathcal{N}(r - \hat{r}, C_r)$, and **(c)** $\xi(r) = \delta(r - \hat{r})$.

Gaussian marginal densities. For a more detailed description for this type of non-Gaussian joint density with Gaussian marginals we refer to our previous research work [10].

For two given Gaussian marginal densities $f_x(x) = \mathcal{N}(\hat{x}, C_x)$ and $f_y(y) = \mathcal{N}(\hat{y}, C_y)$, a family of possible joint densities, depending on the generalized correlation function $\xi(r)$, can be parameterized by

$$f(x, y) = \int_{-1}^{1} \xi(r) \mathcal{N}\left(\begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}, \mathbf{C}(r)\right) dr \ , \qquad (2)$$

where $\xi(r)$ is defined on $r \in [-1, 1]$. The parameterized continuous Gaussian mixture $f(x, y)$ is a valid normalized density function for

$$\xi(r) \geq 0 \ , \qquad \int_{-1}^{1} \xi(r) dr = 1 \ .$$

Three examples for this type of parameterized joint density are depicted in Fig. 3.

B. Gaussian Mixture Marginals

In this section, we generalize these ideas to the parameterization of joint densities with Gaussian mixture marginals. Since Gaussian mixtures consist of the convex sum of Gaussian densities and are known as universal approximators, they are well-suited for nonlinear estimation problems.

Thus, finding a parameterization for the imprecisely known joint density with Gaussian mixture marginals, it is possible to develop a novel filtering technique which is able to cope with both nonlinear system models and nonlinear measurement models in a robust manner. As it was mentioned in the introduction, the challenge of a robust decentralized estimation based on Gaussian mixtures is that the classical correlation coefficient r is not a sufficient measure for the stochastic dependency of Gaussian mixtures. Therefore we define in this section a *generalized correlation* *parameter vector* $\underline{\xi}$ for Gaussian mixtures. Finding bounding densities, which are compatible with all stochastic dependency structures in terms of $\underline{\xi}$, it is possible to derive a robust filtering technique for distributed nonlinear problems.

For the sake of simplicity consider two scalar Gaussian mixture marginals according to

$$f_x(x) = \sum_{i=1}^{m} w_{x,i} \mathcal{N}\left(x - \hat{x}_i, C_{x,i}\right) \quad , \tag{3}$$

$$f_y(y) = \sum_{i=1}^{n} w_{y,i} \mathcal{N}\left(y - \hat{y}_i, C_{y,i}\right) \quad , \tag{4}$$

where $\hat{x}_i$ and $\hat{y}_i$ are the individual means, $C_{x,i}$ and $C_{y,i}$ are the individual variances, and $w_{x,i}$ and $w_{y,i}$ are the individual weighting factors, which must be positive and sum to one.

The weighting factors $w_{x,i}$ and $w_{y,i}$ of the marginals are rearranged into vector form according to

$$\begin{aligned} \underline{w}_x &= \begin{bmatrix} w_{x,1} & \cdots & w_{x,m} \end{bmatrix}^T \quad , \\ \underline{w}_y &= \begin{bmatrix} w_{y,1} & \cdots & w_{y,n} \end{bmatrix}^T \quad . \end{aligned}$$

Given two Gaussian mixture marginal densities $f_x(x)$ and $f_y(y)$, a family of possible joint densities $f(x,y)$ depending on the weighting factors w_{ij} is defined by

$$f(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \mathcal{N}\left(\begin{bmatrix} x - \hat{x}_i \\ y - \hat{y}_j \end{bmatrix}, \mathbf{C}_{ij}(r_{ij}) \right) \quad . \tag{5}$$

To assure that the parameterized Gaussian mixture $f(x,y)$ is a valid normalized density function, the weighting factors w_{ij} of the joint density must be positive and sum to one

$$w_{ij} \geq 0 \quad , \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} = 1 \quad . \tag{6}$$

In addition, the weighting factors of the joint density must satisfy

$$\sum_{j=1}^{n} w_{ij} = w_{x,i} \quad , \qquad \sum_{i=1}^{m} w_{ij} = w_{y,j} \quad , \tag{7}$$

to ensure that the parameterized joint density $f(x,y)$ is a valid joint density for the given marginal density functions $f_x(x)$ and $f_y(y)$.

It is more convenient to rearrange the weighting factors of the joint Gaussian mixture density w_{ij} from matrix form to vector form according to

$$\underline{w} = \begin{bmatrix} w_{11} & \cdots & w_{1n}, & w_{21} & \cdots & w_{2n}, & \cdots & w_{mn} \end{bmatrix}^T$$

As it was shown in our previous work [10], under the conditions (6) and (7) the weighting factors $\underline{w}$ of the joint density $f(x,y)$ can be derived according to

$$\underline{w} = \mathbf{T}_e^{\dagger} \begin{bmatrix} \underline{w}_x \\ \underline{w}_y \\ \underline{\xi} \end{bmatrix} \quad , \tag{8}$$

where $\mathbf{T}_e$ describes a unique transformation of weighting factors for valid joint densities to weighting factors for given

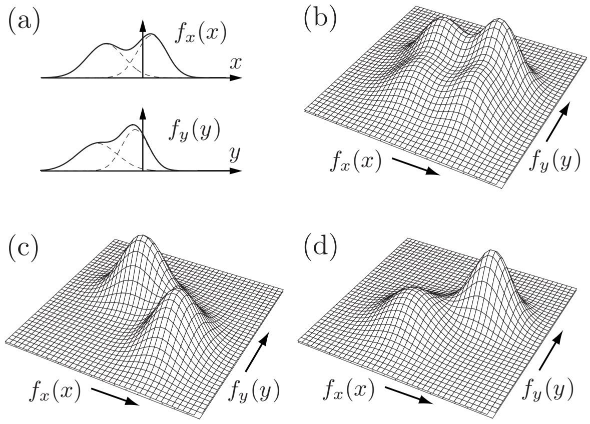

Fig. 4. **(a)** Gaussian mixture marginal densities with two individual components ($w_{x,i} = w_{y,i} = 0.5$), $f_x(x)$ with $\hat{x}_1 = -3$, $\hat{x}_2 = 2$, $C_{x,1} = 2$, $C_{x,2} = 1.6$, and $f_y(y)$ with $\hat{y} = -1$, $\hat{y} = -4$, $C_y = 1.4$, $C_y = 2.1$. Parameterized joint density for various generalized correlation parameter vectors $\underline{\xi}$, **(b)** $\underline{\xi} = 0$, **(c)** $\underline{\xi} = -1$, **(d)** $\underline{\xi} = 1$.

marginal density. The pseudo-inverse is denoted by $\mathbf{T}_e^{\dagger}$. For a more detailed description refer to [10].

Similar to the other types of parameterized joint densities the free parameter vector $\underline{\xi}$ can be regarded as a kind of generalized correlation parameter vector for Gaussian mixtures. Furthermore, the classical correlation coefficients r_{ij} of each Gaussian mixture component is a free parameter. These free parameters need to be specified in order to define the joint density $f(x,y)$ uniquely. Possible joint densities $f(x,y)$ for two given Gaussian mixture marginals $f_x(x)$ and $f_y(y)$ for various parameter vectors $\underline{\xi}$ are depicted in Fig. 4(b)–(d).

IV. Prediction Step (Time Update)

For demonstrating how the parameterized joint densities can be possibly used for a novel filtering technique, a typical application is investigated: Propagation of a given state through a system model. For the sake of simplicity, consider a simple discrete-time dynamic model with the system state $x_k \in \mathbb{R}$, and the system input $u_k \in \mathbb{R}$ according to

$$x_{k+1} = a(x_k, u_k) \quad , \tag{9}$$

where x_k and u_k are random variables represented by the density functions $f_x(x_k)$ and $f_u(u_k)$, respectively.

In the case of precisely known joint density $f_k^e(x_k, u_k)$, the predicted density is given by $f_{k+1}^p(x_{k+1}) =$

$$\int_{\mathbb{R}^2} \delta(x_{k+1} - a(x_k, u_k)) f_k^e(x_k, u_k) dx_k du_k \quad , \tag{10}$$

where $\delta(.)$ denotes the Dirac delta distribution.

However, we assume that the state estimate x_k and the system input u_k are stochastically dependent with a *not precisely known* structure. That means, although the marginal density functions $f_x^e(x_k)$ and $f_u^e(u_k)$ are known, the joint density $f_k^e(x_k, u_k)$ with all its information about the stochastic dependency is unknown. As it can be seen in (10), the knowledge of the joint density $f_k^e(x_k, u_k)$ or at least

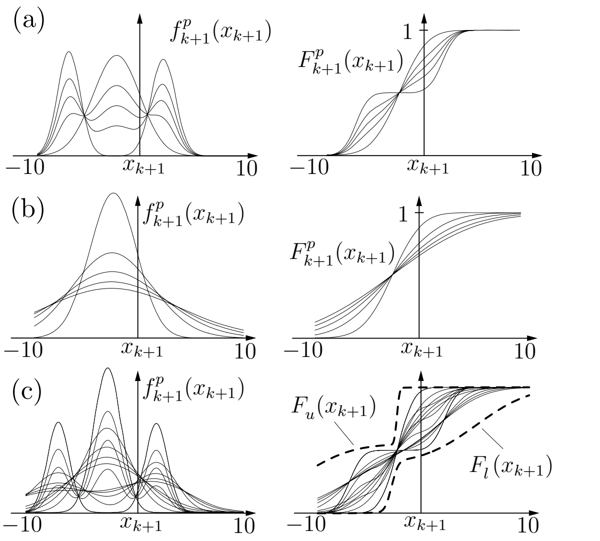

Fig. 5. Predicted density $f^p_{k+1}(x_{k+1})$ and its cumulative distribution function $F^p_{k+1}(x_{k+1})$ for variation of the generalized correlation parameters of $f^e_k(x_k, u_k)$. **(a)** $\alpha_{ij} = -1$, $\underline{\xi} = -1 \ldots 1$. **(b)** $\underline{\xi} = 1$, $\alpha_{ij} = -1 \ldots 1$. **(c)** Various members of the family of possible predicted densities and their bounding densities, $F_l(x_{k+1})$ and $F_u(x_{k+1})$.

its parameterization in terms of a correlation parameter is essential for deriving prediction results.

The goal is to find a bounding density for the prediction result f^p_{k+1} of all possible joint densities $f^e_k(x_k, u_k)$. Thus, based on structural assumptions concerning the stochastic dependency the unknown prior density $f^e_k(x_k, u_k)$ can be parameterized by

$$f^e_k = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \int_{-1}^{1} \xi_{ij}(r) \mathcal{N}\left(\begin{bmatrix} x_k - \hat{x}^e_{k,i} \\ u_k - \hat{u}^e_{k,j} \end{bmatrix}, \mathbf{C}^e_{ij}(r) \right) dr,$$

where $\xi_{ij}(r)$ and $\underline{\xi}$ (affecting the calculation of w_{ij}) are the generalized correlation function and the generalized correlation parameter vector, respectively. The individual joint covariance matrix between the i-th component of $f^e_x(x_k)$ and j-th component of $f^e_u(u_k)$ is given by

$$\mathbf{C}^e_{ij}(r) = \begin{bmatrix} C_{x,i} & r\sqrt{C_{x,i}C_{u,j}} \\ r\sqrt{C_{x,i}C_{u,j}} & C_{u,j} \end{bmatrix} \quad ,$$

where r denotes the *classical* correlation parameter.

By means of the parameterized prior joint density $f^e_k(x_k, u_k)$ the resulting predicted density $f^p_{k+1}(x_{k+1})$ can be described in terms of a generalized correlation parameter vector $\underline{\xi}$ and a generalized correlation function $\xi(r)$. This parameterized density function $f^p_{k+1}(x_{k+1})$ describes a whole family of possible prediction results. The following example clarifies these results.

EXAMPLE 1 For the sake of simplicity and brevity we consider a linear state space model according to

$$x_{k+1} = A_k x_k + B_k u_k \quad ,$$

where the two random variables x_k and u_k are represented by Gaussian mixtures with two components, visualized in Fig. 4 (a), i.e., $f^e_k(x_k) \equiv f_x(x)$ and $f^e_k(u_k) \equiv f_y(y)$.

Although more complex correlation functions $\xi_{ij}(r)$ for the parameterization of the prior density f^e_k can be chosen, for simplicity this function is given by

$$\xi_{ij}(r) = \delta(r - \alpha_{ij}) \quad , \tag{11}$$

where α_{ij} denotes a specific *classical* correlation coefficient.

The predicted density f^p_{k+1} for various generalized correlation parameter vectors $\underline{\xi}$ and α_{ij} is depicted in Fig. 5(a)–(c). These figures clearly show that the prediction result f^p_{k+1} strongly depends on the stochastic dependency, however can be described by a generalized correlation parameter.

It is obvious that the parameterized predicted distribution function F^p_k can be bounded by bounding distribution functions $F_u(x_{k+1})$ (upper bound) and $F_l(x_{k+1})$ (lower bound), depicted in Fig. 5(c). Furthermore, it can be said that once a representation of such bounding densities is found, a filtering technique can be derived, which can cope with *nonlinear models* and is robust against imprecisely known stochastic dependencies. The actual calculation of the bounding densities is left for future research work.

V. FILTER STEP (MEASUREMENT UPDATE)

In this section we take up the example mentioned in the introduction: Decentralized self-localization of a sensor-actuator-network. For illustration purposes, we consider just two sensor nodes and assume that the relative distance measurement $\hat{y}_k$ is related *non*linearly to their positions. Thus, the measurement equation is given by

$$\hat{y}_k = h_k(\underline{x}_k) + v_k \quad ,$$

where $\underline{x}_k = [x^{(1)}_k, x^{(2)}_k]^T$ are the estimated sensor positions and v_k represents the measurement uncertainty.

In the case of precisely known joint density $f^p_k(x^{(1)}_k, x^{(2)}_k)$, the posterior density f^e_k can be easily calculated by

$$f^e_k(\underline{x}_k) = c^e_k \, f^L(\hat{y}_k - h_k(\underline{x}_k)) \, f^p_k(x^{(1)}_k, x^{(2)}_k) \quad , \tag{12}$$

where c^e_k is a normalization constant.

However, we assume an *imprecisely* known stochastic dependency between the individual estimated positions. That means, although the marginal density functions $f^p_k(x^{(1)}_k)$ and $f^p_k(x^{(2)}_k)$ are known, the joint density $f^p_k(x^{(1)}_k, x^{(2)}_k)$ with its information about the stochastic dependency is unknown. Unfortunately, the knowledge of the joint density $f^p_k(x^{(1)}_k, x^{(2)}_k)$ or at least its parameterization in terms of a correlation parameter is essential for deriving estimated densities f^e_k by means of (12).

Thus, similar to the time update, the *imprecisely known* joint density $f^p_k(x^{(1)}_k, x^{(2)}_k)$ can be parameterized by a joint density according to

$$f^p_k = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \int_{-1}^{1} \xi_{ij}(r) \mathcal{N}\left(\begin{bmatrix} x^{(1)}_k - \hat{x}^{e,(1)}_{k,i} \\ x^{(2)}_k - \hat{x}^{e,(2)}_{k,j} \end{bmatrix}, \mathbf{C}^p_{ij}(r) \right) dr,$$

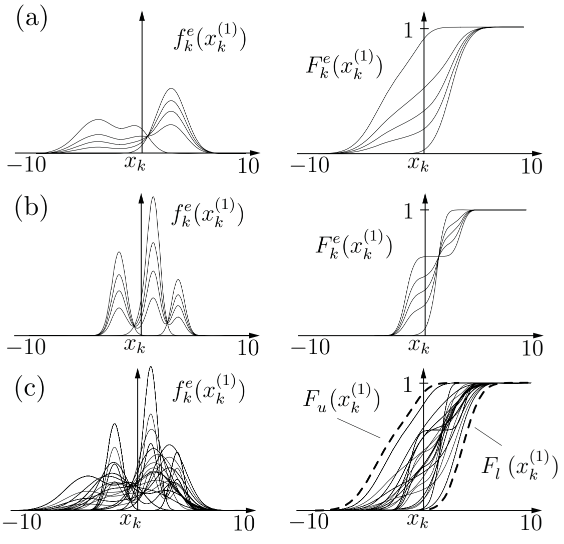

Fig. 6. Estimated density $f_k^e(x_k^{(1)})$ and its cumulative distribution function $F_k^e(x_k^{(1)})$ for variation of the generalized correlation parameters of $f_k^p(x_k^{(1)}, x_k^{(2)})$. **(a)** $\alpha_{ij} = -1$, $\underline{\xi} = -1 \dots 1$. **(b)** $\alpha_{ij} = 1$, $\underline{\xi} = -1 \dots 1$. **(c)** Various members of possible estimated densities and their bounding densities, $F_l(x_k^{(1)})$ and $F_u(x_k^{(1)})$.

where $\xi_{ij}(r)$ and $\underline{\xi}$ (affecting the calculation of w_{ij}) are the generalized correlation function and the generalized correlation vector, respectively. The following example illustrates the results.

EXAMPLE 2 For the sake of simplicity we consider a linear measurement equation given by

$$\hat{y} = x_k^{(1)} + x_k^{(2)} + v_k \ ,$$

where the two random variables $x_k^{(1)}$ and $x_k^{(2)}$ are represented by Gaussian mixtures with two components, visualized in Fig. 4 (a), i.e., $f_k^p(x_k^{(1)}) \equiv f_x(x)$ and $f_k^p(x_k^{(2)}) \equiv f_y(y)$.

The estimated density $f_k^e(x^{(1)})$ for various generalized correlation parameter vectors $\underline{\xi}$ and α_{ij} is depicted in Fig. 6 (a)–(b). These figures clearly show that the estimation result strongly depends on the stochastic dependency, however can be parameterized in terms of a generalized correlation parameter $\underline{\xi}$ and a generalized correlation function $\xi(r)$.

It is ovious that the parameterized estimated distribution function $F_k^e(x^{(1)})$ can be bounded by means of bounding distribution functions $F_u(x_k^{(1)})$ (upper bound) and $F_l(x_k^{(1)})$ (lower bound), depicted in Fig. 6 (c).

VI. CONCLUSIONS AND FUTURE WORK

This paper focuses on the parameterization of different types of joint densities with both Gaussian marginals and Gaussian mixture marginals. It is shown that by assuming a specific stochastic dependency structure these joint densities contain all information about this dependency in terms of a *generalized correlation parameter* $\underline{\xi}$ and/or a *generalized correlation function* $\xi(r)$. Unlike the *classical* correlation coefficient r the generalized correlation parameter $\underline{\xi}$ and function $\xi(r)$ is a sufficient measure for the stochastic dependency between two random variables represented by Gaussian mixtures.

Depending on these correlation parameters, detailed prediction results and measurement results are presented. Furthermore, it is shown that there could exist bounding densities containing all possible joint densities characterized by the generalized correlation parameter vector $\underline{\xi}$ and the generalized correlation function $\xi(r)$.

To find such bounding densities fulfilling the stochastic dependency constraints is left for future research. A possible direction for finding bounding densities can be found in [11]. Once such a bounding density is found, the derivation of a filtering technique, which can cope with *nonlinear models* and is robust against imprecisely known stochastic dependencies is possible.

VII. ACKNOWLEDGEMENTS

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